TENSOR PRODUCTS OF CRYSTALS AND REP'S.

$\pi_1, \pi_2 : G \to GL(V_i) \quad i = 1, 2$

$\pi_1 \otimes \pi_2 : G \to GL(V_1 \otimes V_2)$

MAY OR MAY NOT BE IRREDUCIBLE.

FOR $G = GL(n, \mathbb{C})$ IRREDUCIBLES ARE PARAMETRIZED BY PARTITIONS OR SLIGHTLY MORE GENERALLY DOMINANT WEIGHTS.

PARTITION: $\lambda = (\lambda_1, \lambda_2, \dots)$ (EVENTUALLY ZERO)

$\lambda_i \geq 0 \qquad \lambda_1 \geq \lambda_2 \geq \dots$

LENGTH(λ) = LARGEST i SUCH THAT $\lambda_i \neq 0$

FOR IRREPS OF $GL(n)$, EVERY PARTITION OF LENGTH $\leq n$ INDEXES A REP'N π_λ.

IRREDUCIBLE

$$\chi_{\pi_\lambda}\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = s_\lambda(t_1, \dots, t_n).$$

THEOREM: LET π_{ST} = STANDARD REP'N

$$\pi_{(1,0,\cdots,0)} : GL(n) \to GL(n)$$
$$g \rightsquigarrow \pi(g) = g$$

IF λ IS A PARTITION OF LENGTH $\leq n$

$k = |\lambda| = \sum \lambda_i$

WE CAN FIND (SOMETIMES SEVERAL) COPIES OF π_λ INSIDE

$$\otimes^k \pi_{\text{STANDARD}} \circlearrowleft \otimes^k \mathbb{C}^n$$

YOU HAVE COMMUTING ACTIONS OF $GL(n, \mathbb{C})$ AND S_k ON $\otimes^k \mathbb{C}^n$

$$\underbrace{g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k}$$

$\sigma \in S_k$ $\quad$ $\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$.

MORE PRECISE RESULT: λ ALSO PARAMETRIZES AN IRREP OF S_k

$$\to \quad \bigotimes^k \mathbb{C}^n \cong \bigoplus \pi_\lambda^{GL_n} \otimes \pi_\lambda^{S_k}.$$

SCHUR-WEYL DUALITY.

$GL(n) \times S_k$

λ A PARTITION OF k OF LENGTH $\leq n$

IF WE CONSIDER $\lambda \in \mathbb{Z}^n$

$\lambda_1 \geq \cdots \geq \lambda_n$

BUT ALLOW $\lambda_n < 0$ WE OBTAIN THE NOTION OF A <u>DOMINANT WEIGHT</u>

$$\pi_\lambda \otimes \det^k = \pi_{(\lambda_1 + k, \cdots, \lambda_n + k)}$$

k CAN BE POSITIVE OR NEGATIVE

SO WE CAN OBTAIN π_λ WHERE λ IS ANY DOMINANT WEIGHT BY STARTING WITH π_μ (μ A PARTITION $\mu_i \geq 0$)

TENSORING WITH A POSSIBLY NEGATIVE POWER OF THE DETERMINANT.

FOR $GL(3)$

$$S_k \times GL(n, \mathbb{C}) \xrightarrow{i} GL(\otimes^k \mathbb{C}^n)$$

EXAMPLES: GL(3)

GL(3) HAS TWO 3-DIM'L IRREPS.

$\pi_{(1,0,0)}$ $\qquad \Delta_{(1,0,0)} = t_1 + t_2 + t_3$

$\pi_{(1,1,0)}$ $\qquad \Delta_{(1,1,0)} = t_1 t_2 + t_1 t_3 + t_2 t_3$

$$\wedge^2 \pi_{(1,0,0)} = \pi_{(1,1,0)}$$

$$\underbrace{\vee^2}_{\text{SYMMETRIC SQUARE}} \pi_{(1,0,0)} = \pi_{(2,0,0)} = t_1^2 + t_2^2 + t_3^2 + t_1 t_2 + t_1 t_3 + t_2 t_3$$

$\pi_{(2,1,0)}$ HAS DIMENSION 8

$$t_1^2 t_2 + t_1^2 t_3 + t_2^2 t_1 + t_2^2 t_3 + t_3^2 t_1 + t_3^2 t_2 + 2 t_1 t_2 t_3.$$

$\mathbb{B}_\lambda$ = CRYSTAL OF π_λ

= ALL SSYT OF SHAPE λ

ENTRIES IN $\{1, 2, \ldots, n\}$

SSYT OF SHAPE λ IS A FILLING OF YOUNG DIAGRAM BY INTEGERS

ROWS WEAKLY INCREASING

COLS STRICTLY INCREASING.

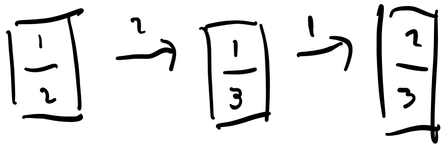
$$\mathrm{wt} : \mathcal{B}_\lambda \to \mathbb{Z}^n$$

$\mathrm{wt}(T) = \lambda$ MEANS T HAS λ_i ENTRIES EQUAL TO i.

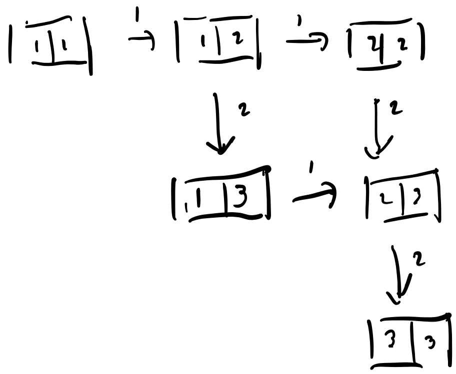
$$e_i, f_i : \mathcal{B}_\lambda \to \mathcal{B}_\lambda \cup \{0\}$$

$$\underset{x}{\bullet} \xrightarrow{\ i\ } \underset{y}{\bullet}$$ MEANS $f_i(x) = y$, $e_i(y) = x$.

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \qquad \mathcal{B}_{(1,0,0)}$$

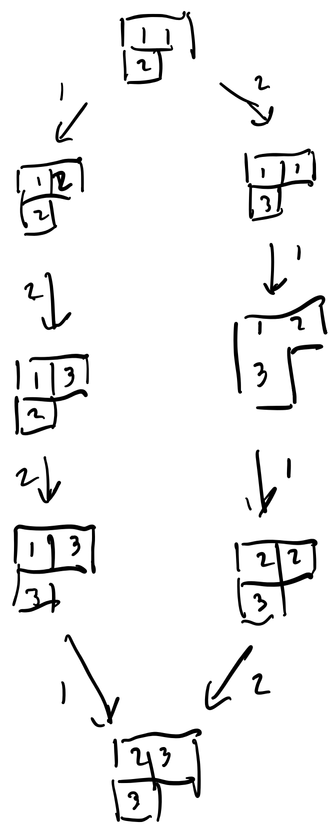


$\mathcal{B}_{(1,1,0)}$



$\mathcal{B}_{(2,0,0)}$

$\mathcal{B}_{(2,1,0)}$



$$s_{(2,1,0)} = t_1^2 t_2 + t_1^2 t_3 + t_2^2 t_1 + t_2^2 t_3 + t_3^2 t_1 + t_3^2 t_2 + 2t_1 t_2 t_3 .$$

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TENSOR PRODUCTS OF CRYSTALS CAN BE DEFINED.

THEOREM: IF λ IS A PARTITION OF LENGTH $\leq n$ AND $|\lambda| = k$

LET $\mathcal{B} = \mathcal{B}_{(1,0,0)}$ BE STANDARD CRYSTAL

$$\mathcal{B}: \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \rightarrow \cdots \xrightarrow{n-1} \boxed{n}$$

BY TENSORING $\mathcal{B}$ TOGETHER k TIMES WE CAN EXTRACT A CRYSTAL $\mathcal{B}_\lambda$ TO BE IDENTIFIED WITH THE "CRYSTAL OF TABLEAUX".

$$\underset{(1,0,0)}{\overset{3}{\Pi}} \otimes \underset{(1,0,0)}{\overset{3}{\Pi}} = \underset{(2,0,0)}{\overset{6}{\Pi}} \oplus \underset{(1,1,0)}{\overset{3}{\Pi}}$$

$n = 3$ $$\underbrace{\mathbb{C}^n \otimes \mathbb{C}^n}_{n^2} = \underbrace{\vee^2 \mathbb{C}^n}_{\frac{1}{2}n(n+1)} \oplus \underbrace{\wedge^2 \mathbb{C}^n}_{\frac{1}{2}n(n-1)}$$

$$x \otimes y = \tfrac{1}{2}(x \otimes y + y \otimes x) + \tfrac{1}{2}(x \otimes y - y \otimes x)$$

$$\overset{3}{\pi_{(1,0,0)}} \otimes \overset{3}{\pi_{(1,1,0)}} = \overset{8}{\pi_{(2,1,0)}} \oplus \underset{\det}{\overset{1}{\pi_{(1,1,1)}}}$$

$$\pi(6,5,5) = \det^5 \otimes \pi_{(1,0,0)}$$

RULE FOR TENSOR PRODUCT OF CRYSTALS WAS FOUND BY KASHIWARA - NAKASHIMA. COMING FROM THEORY OF QUANTUM GROUPS. FROM OUR POINT OF VIEW THIS IS JUST A COMBINATORIAL DEFINITION.

$\varphi_i(x)$ = # OF TIMES WE CAN APPLY f_i TO x

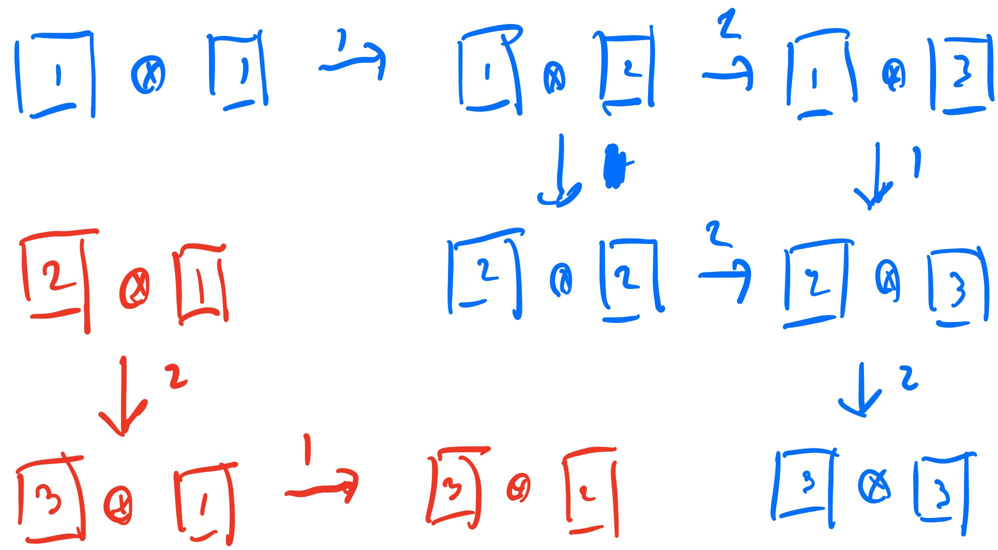
$\varepsilon_i(x)$ = # OF TIMES WE CAN APPLY e_i TO X

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{IF } \varphi_i(y) \leq \varepsilon_i(x) \\ x \otimes f_i(y) & \text{IF } \varphi_i(y) > \varepsilon_i(x) \end{cases}$$

IF $f_i(x) = 0$ THEN $f(x) \otimes y$ IS INTERPRETED TO MEAN 0.

$$\mathbb{B} = \mathbb{B}_{(1,0,0)}$$

COMPUTE $\mathbb{B} \otimes \mathbb{B}$.



$$f_i(x\otimes y) = \begin{cases} f_i(x)\otimes y & \text{IF } \varphi_i(y) \le \varepsilon_i(x) \\ x\otimes f_i(y) & \text{IF } \varphi_i(y) > \varepsilon_i(x) \end{cases}$$

$f_1(\boxed{1}\otimes\boxed{1})$

$\varphi_1(\boxed{1}) = 1$ SINCE WE CAN APPLY f_1 ONCE

$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$

$\varepsilon_1(\boxed{1}) = 0$ CAN'T APPLY e_1 TO $\boxed{1}$.

SO $\varphi_1(\boxed{1}) > \varepsilon_1(\boxed{1})$

$$\varphi_2(\boxed{1}) = \varepsilon_2(\boxed{1}) = 0$$

FIRST CASE $f_2(\boxed{1} \otimes \boxed{1}) =$

$$\underbrace{f_2(\boxed{1})}_{\text{ZERO}} \otimes \boxed{1} = 0$$

$$\overset{x}{\boxed{2}} \otimes \overset{y}{\boxed{1}}$$

$$\varphi_1(\boxed{1}) = 1 \qquad \varepsilon_1(\boxed{2}) = 1$$

$$e_1(\boxed{2}) = \boxed{1}$$

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{IF } \varphi_i(y) \le \varepsilon_i(x) \\ x \otimes f_i(y) & \text{IF } \varphi_i(y) > \varepsilon_i(x) \end{cases}$$

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2}$$

FIRST CASE

$$f_1(\boxed{2} \otimes \boxed{1}) = \underbrace{f_1(\boxed{2})}_{\text{ZERO}} \otimes \boxed{1} = 0$$

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

$$\overset{x}{\boxed{2}} \otimes \overset{y}{\boxed{1}}$$

$$\varphi_2\left(\boxed{1}\right) = 0$$

$$\varepsilon_2\left(\boxed{1}\right) = 0$$

FIRST CASE $f_2\left(\boxed{2} \otimes \boxed{1}\right) = f_2\left(\boxed{2}\right) \otimes \boxed{1}$

$$= \boxed{3} \otimes \boxed{1}$$

EXERCISE: DECOMPOSE

$$\mathcal{B}_{(1,0,0)} \otimes \mathcal{B}_{(1,1,0)} = \mathcal{B}_{(2,1,0)} \oplus \mathcal{B}_{(1,1,1)}$$

$$\mathcal{B}_{(1,1,1)} = \boxed{1 \,|\, 2 \,|\, 3}$$

CRYSTAL of det.

$\boxed{1} \otimes \boxed{1}\boxed{2}$ $\quad$ $\boxed{1} \otimes \boxed{1}\boxed{3}$ $\quad$ $\boxed{1} \otimes \boxed{2}\boxed{3}$

$\boxed{2} \otimes \boxed{1}\boxed{2}$ $\quad$ $\boxed{2} \otimes \boxed{1}\boxed{3}$ $\quad$ $\boxed{2} \otimes \boxed{2}\boxed{3}$

$\boxed{3} \otimes \boxed{1}\boxed{2}$ $\quad$ $\boxed{3} \otimes \boxed{1}\boxed{3}$ $\quad$ $\boxed{3} \otimes \boxed{2}\boxed{3}$

COMPUTE ρ_1, ρ_2 FOR THESE TENSORS. $\quad 3 \otimes 3 = 8 \oplus 1$.